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Short Note

A new minimum storage Runge–Kutta scheme for computational acoustics

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Abstract

A new fourth-order six-stage Runge–Kutta numerical integrator that requires $2N$-storage (N is the number of degrees of freedom of the system) with low dissipation and dispersion and a relatively large stability interval is proposed. These features make it a suitable time advancing method for solving wave propagation problems in Computational Acoustics. Some numerical experiments are presented to show the favourable behaviour of the new scheme as compared with the LDD46 and LDD25 methods proposed by Stanescu and Habashi [J. Comput. Phys. 143 (1998) 674] and the standard fourth order Runge–Kutta method.

AMS: 65L05; 65M20

Keywords: Time advancing methods; Low storage RK schemes; Dissipation and dispersion

1. Introduction

Explicit Runge–Kutta (RK) schemes with *low dissipation and dispersion* (*LDD*) and large stability intervals are among the most popular time advancing schemes for computational acoustics. Thus, Hu, Hussaini and Manthey proposed in [7] several low-dissipation and low-dispersion Runge–Kutta schemes (LDDRK) whose stability functions were constructed to minimize in some sense the dispersion and dissipation errors while maintaining large intervals of stability, showing that they were very efficient for wave propagation problems. Other RK schemes with the same aim have been proposed by Mead and Renaut [9] in the context of pseudospectral discretizations. A large number of alternative proposals for problems with different kinds of oscillating solutions have been given, among them those of Simos and coworkers (see [1] and [10]) and the present authors [2].

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Since standard problems in Computational Acoustics usually have large memory requirements, to improve the efficiency of time advancing numerical schemes for these problems, some authors like Williamson [13], Carpenter, Kennedy and Lewis [4,8], Stanescu and Habashi [11] and the present authors [3] have proposed special RK schemes that can be written using minimum storage (i.e. $2N$-storage, where N is the dimension of the first order differential system). Clearly, a suitable combination of good stability properties, high order, low dissipation and dispersion and low storage may lead to a near optimal scheme for the type of problems under consideration.

The aim of this note is to propose a new fourth-order RK scheme with six stages which attempts to be optimal in the above sense. It is shown that the new scheme compares favourably with the two similar schemes LDD46 and LDD25 given by Stanescu and Habashi in [11] that were constructed from stability functions that minimize the dispersion and dissipation errors for linear wave propagation proposed by Hu et al. [7].

The paper is organized as follows: In Section 2 we introduce the class of minimum storage schemes under consideration and the criteria used to determine the available parameters in order to get the desired order, stability and dissipation and dispersion properties. In Section 3 some numerical experiments are presented comparing the behaviour of the new scheme with two optimal low-dissipation and low-dispersion schemes with five and six stages of Stanescu and Habashi [11] denoted by LDD25 and LDD46 as well as the popular fourth order RK method [6, p. 138].

2. The choice of an optimal scheme in the class of minimum storage RK schemes

Suppose the time evolution differential system written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = F(U(t)), \quad t \geqslant 0, \tag{1}$$

where t is the time, $U : \mathbb{R}^+ \to \mathbb{R}^N$ is the state vector of the PDE solution at the spatial grid points and F is the operator containing the discretization of spatial derivatives. Since we are concerned with wave phenomena with smooth solutions, the linear part of the right hand side function (1) will be assumed to have purely imaginary eigenvalues and therefore the stability intervals under consideration of the numerical schemes to be given below will be on the imaginary axis (see [7], Section 2).

The $2N$-storage RK scheme that advances the state vector U^n at t_n to U^{n+1} at $t_{n+1} = t_n + \Delta t$ uses two N-registers in which the following two N-vectors u_i and F_i are successively stored

$$\begin{array}{ll} u_1 = U^n, & F_1 = F(u_1), \\ u_2 = u_1 + \Delta t b_1 F_1, & F_2 = F(u_2 + \Delta t \gamma_1 F_1), \\ \vdots & \vdots \\ u_s = u_{s-1} + \Delta t b_{s-1} F_{s-1}, & F_s = F(u_s + \Delta t \gamma_{s-1} F_{s-1}), \\ u_{s+1} = u_s + \Delta t b_s F_s, & \end{array} \tag{2}$$

and the state vector U^{n+1} at t_{n+1} is given by $U^{n+1} = u_{s+1}$. Here $b_1, \ldots, b_s$ and $\gamma_1, \ldots, \gamma_{s-1}$ are $2s-1$ real parameters that define the minimum storage RK scheme. Note that our $2N$-storage algorithm is not equivalent to the low-storage algorithm used by Stanescu and Habashi [11].

This algorithm (2) can be written equivalently as the s-stage Runge–Kutta scheme

$$\left.\begin{array}{l} F_1 = F(U^n), \\ F_i = F\left(U^n + \Delta t \sum_{j=1}^{i-1} b_j F_j + \Delta t \gamma_{i-1} F_{i-1}\right), \ (i = 2, \ldots, s) \\ U^{n+1} = U^n + \Delta t \sum_{i=1}^{s} b_i F_i, \end{array}\right\} \tag{3}$$

and it is usually specified by the lower triangular matrix $A \in \mathbb{R}^{s\times s}$ and the vector $b \in \mathbb{R}^s$ given by

$$A = \begin{pmatrix} 0 & & & & \\ b_1+\gamma_1 & 0 & & & \\ b_1 & b_2+\gamma_2 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ b_1 & b_2 & \dots & b_{s-1}+\gamma_{s-1} & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{s-1} \\ b_s \end{pmatrix}. \tag{4}$$

Hence the $2N$-storage time advancing method (2) is equivalent to the s-stage explicit RK scheme (3) with (A, b) defined by (4). Observe that the set of all s-stage minimum storage RK methods (2) or (4) depend on $(2s-1)$ parameters whereas a general s-stage explicit RK scheme depends on $s(s-1)/2+s$ parameters.

In the following we will restrict our considerations to $2N$-storage methods (3) with $s = 6$ stages. By introducing the vectors $e = (1,\dots,1)^{\mathrm{T}} \in \mathbb{R}^6$ and $c = Ae \in \mathbb{R}^6$, the eight conditions on the coefficients (A, b) for order four (see [6], p. 153) can be written in the form

$$\begin{aligned} &b^{\mathrm{T}}e = 1, \quad b^{\mathrm{T}}c = 1/2, \quad b^{\mathrm{T}}Ac = 1/6, \quad b^{\mathrm{T}}A^2c = 1/24, \\ &b^{\mathrm{T}}c^2 = 1/3, \quad b^{\mathrm{T}}c^3 = 1/4, \quad b^{\mathrm{T}}(c\cdot Ac) = 1/8, \quad b^{\mathrm{T}}Ac^2 = 1/12. \end{aligned} \tag{5}$$

Here the dot product is the componentwise product and $c^k = c\cdot\ldots\cdot c$.

The stability as well as the dissipation–dispersion properties of (3) or (4) are concerned with their behaviour for linear scalar problems $F(U) = \lambda U$ with λ on the imaginary axis. We apply the method (3) to the linear scalar test equation $U' = F(U) \equiv \mathrm{i}\omega U$, $\omega \in \mathbb{R}$, $\mathrm{i} = \sqrt{-1}$, obtaining

$$U^{n+1} = R(\mathrm{i}\omega\Delta t)U^n,$$

where $R(z)$ is the stability function of the method (3) given by the polynomial

$$R(z) = 1 + \sum_{j=1}^{6}(b^{\mathrm{T}}A^{j-1}e)z^j.$$

The fourth order conditions (5) imply that $b^{\mathrm{T}}A^{j-1}e = 1/j!$, $j = 1,\dots,4$, and introducing the real parameters given by $\beta_5 = 5!(b^{\mathrm{T}}A^3c)$, $\beta_6 = 6!(b^{\mathrm{T}}A^4c)$, the stability function can be written in the form

$$R(z) = R(z;\beta_5,\beta_6) = \sum_{j=0}^{4}\frac{z^j}{j!} + \frac{\beta_5}{5!}z^5 + \frac{\beta_6}{6!}z^6. \tag{6}$$

Now our goal is to choose the real parameters β_5, β_6 to satisfy the following requirements:

- Maximize the interval of absolute stability $[0, S]$ of (6) with $S = S(\beta_5,\beta_6)$ defined by

$$S(\beta_5,\beta_6) = \max\{\delta \geqslant 0; |R(\mathrm{i}v;\beta_5,\beta_6)| \leqslant 1, \forall v \in [0,\delta]\}$$

- Maximize a measure $L = L(\beta_5,\beta_6)$ associated to both dispersion $\phi(v) = v - \arg(R(\mathrm{i}v))$ and dissipation $d(v) = 1 - |R(\mathrm{i}v)|$ errors of (6) given by

$$L(\beta_5,\beta_6) = \max\{\lambda \geqslant 0; |d(v)| < |\phi(v)| \leqslant 1.25\times 10^{-3}, \forall v \in [0,\lambda]\}.$$

(Here we have taken 1.25×10^{-3} as an upper bound of dissipation and dispersion errors in a similar way to [7] where 10^{-3} has been taken).

It turns out that $S(\beta_5,\beta_6)$ and $L(\beta_5,\beta_6)$ cannot be maximized for the same set of values of β_5 and β_6. Then we have maximized the continuous function $(1/2)S(\beta_5,\beta_6) + (1/2)L(\beta_5,\beta_6)$, which amounts to consideration of both functions with the same weight, in the compact set $(\beta_5,\beta_6) \in [0,1]\times[0,1]$. Observe

Table 1
Coefficients β_5, β_6, L and S for the stability functions

Stability function	β_5	β_6	L	S
$R_{\mathrm{hu}}(z)$	0.937206	0.951415	1.77	1.75
$R_{\mathrm{new}}(z)$	0.9424	0.683201	1.19	3.82

that for $\beta_5 = \beta_6 = 0$, $R(z; 0, 0)$ is the stability function of a four stage RK method with order four (for linear equations) and $S(0, 0) \simeq 2.75$, and for $\beta_5 = \beta_6 = 1$, $R(z; 1, 1)$ is the stability function of a six stage RK method with order four (for linear equations) and $S(1, 1) = 0$. The search of a maximum of $(1/2)S(\beta_5, \beta_6) + (1/2)L(\beta_5, \beta_6)$ in the above set has been carried out numerically obtaining the values $\beta_5^* = 0.9424, \beta_6^* = 0.683201$. In Table 1 we include the values of the parameters β_5, β_6 of our six-stage fourth order stability function $R_{\mathrm{new}}(z)$ and the corresponding values of the stability function $R_{\mathrm{hu}}(z)$ of Hu et al. [7], as well as the values of S and L for both stability functions. Observe that, according to (6), our values of β_5, β_6 have been scaled by the corresponding factorials in contrast with the values given in Table 2 of [7].

It should be remarked that other alternative measures of the dissipation and dispersion errors have been used in the literature (see [2,12]). Thus, Van der Houwen and Sommeijer [12] consider the order of dissipation k of explicit RK methods that is defined as the maximum integer k such that $d(v) = \mathcal{O}(v^{k+1})$, $v \to 0$ and similarly for the dispersion. These local measures are reliable for $v = \omega\Delta t$ in a interval $[0, \epsilon]$ with ϵ sufficiently small, but such ϵ could be very small for practical purposes. Hence we have preferred to take the above semi-local measure. On the other hand Hu et al. [7] use as a measure the quantity

$$\int_0^{\mathrm{T}} |R(\mathrm{i}v) - \exp(\mathrm{i}v)|^2 \mathrm{d}v,$$

with a suitable selected value of T. In this case, since

$$|R(\mathrm{i}v) - \exp(\mathrm{i}v)|^2 = d(v)^2 + (1 - d(v))\phi(v)^2 + \mathcal{O}(\phi^4),$$

it is clear that both dissipation and dispersion, with almost the same weight, are taken into account in the above integral sense.

In Fig. 1 we plot the dissipation error $d(v)$ as a function of v for both stability functions R_{hu} and R_{new} in the interval $v \in [0, 2]$. Clearly the behaviour of d_{new} is superior to d_{hu} in the interval $[0, 1.6]$. Further it can be

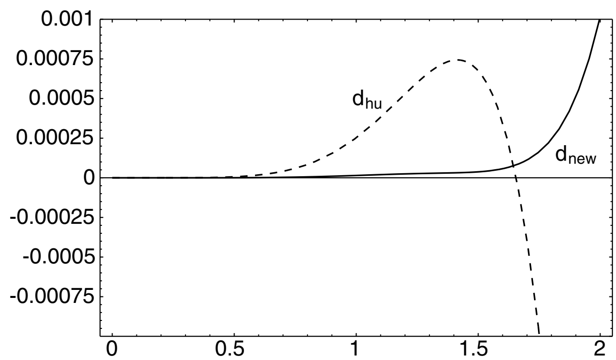

Fig. 1. Dissipation error $d(v)$.

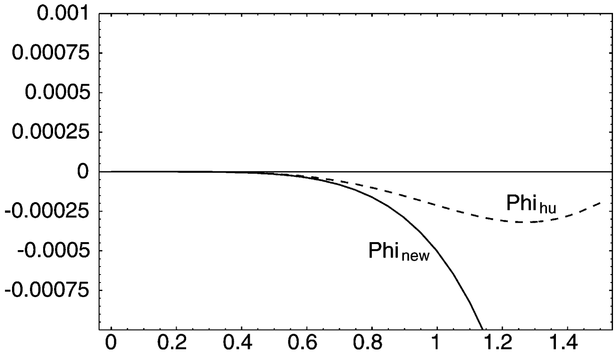

Fig. 2. Dispersion error $\phi(v)$.

seen that d_{hu} is monotonic decreasing for all $v \geqslant 1.6$ whereas d_{new}, which is at first an increasing function, vanishes again at the stability limit $v = 3.82$. In conclusion, $|d_{\text{hu}}(v)| > |d_{\text{new}}(v)|$ for $v \geqslant 1.75$ up to $v = 3.82$.

On the other hand it follows from Fig. 2, that the dispersion error $\phi_{\text{hu}}(v)$ is superior to the dispersion error of the new method ϕ_{new}. Only in an interval $[0, v^*]$ with $v^* < 1$ we get a similar behaviour.

Next, after the above choice of the stability function, we must select the eleven coefficients $b_i, (i = 1, \dots, 6)$ and $\gamma_i, (i = 1, \dots, 5)$ satisfying the eight order conditions (5) of order four and the two additional conditions

$$b^{\mathrm{T}} A^3 c = \frac{\beta_5}{5!}, \qquad b^{\mathrm{T}} A^4 c = \frac{\beta_6}{6!}. \tag{7}$$

This system of ten (nonlinear) equations with eleven free parameters *possesses* an infinite set of solutions. This degree of freedom has been used to determine numerically a solution of (5) and (7) taking into account the following natural requirements:

- $|b_i| \leqslant 2, \quad i = 1, \dots, n,$ i.e. the coefficients of the underlying quadrature formula of (3): $\int_0^{\Delta t} f(t)\mathrm{d}t = \Delta t \sum_{i=1}^{s} b_i f(c_i \Delta t)$ be moderately sized.
- The components c_i $(i = 1, \dots, n)$ of the vector $c = Ae$ satisfy $c_i \neq c_j$ for all $i \neq j$ and $c_i \in [0, 1]$, i.e. the nodes $c_i \Delta t$ of the underlying quadrature formula of (3) be in the interval $[0, \Delta t]$.
- Minimize the ℓ_2-norm of the leading term of the local error of method (3), ($\|\tau^{(5)}\|_2$) defined by $\|\tau^{(5)}\|_2^2 = \sum_j |C_{5,j}|^2$ where $C_{5,j}$ are the coefficients of the elementary differentials of order five in the local error expansion in powers of the step size Δt (see [6], Section II.3).

The values of the selected parameters, for which $\|\tau^{(5)}\|_2 = 2.86 \times 10^{-3}$, are given in Table 2.

Table 2
Coefficients of the new $2N$-storage LDD46

$c_1 = 0$	$b_1 = 0.10893125722541$	$\gamma_1 = 0.17985400977138$
$c_2 = 0.28878526699679$	$b_2 = 0.13201701492152$	$\gamma_2 = 0.14081893152111$
$c_3 = 0.38176720366804$	$b_3 = 0.38911623225517$	$\gamma_3 = 0.08255631629428$
$c_4 = 0.71262082069639$	$b_4 = -0.59203884581148$	$\gamma_4 = 0.65804425034331$
$c_5 = 0.69606990893393$	$b_5 = 0.47385028714844$	$\gamma_5 = 0.31862993413251$
$c_6 = 0.83050587987157$	$b_6 = 0.48812405426094$	

3. Numerical experiments

In order to test the behaviour of the new LDDRK method, we present here some numerical results obtained for several model problems associated with the wave propagation. The new scheme has been compared with

- The $2N$-storage scheme with six stages and order four of Stanescu and Habashi [11] denoted by LDD46 that was constructed on the basis of the optimized six-stages stability function of Hu et al. [7].
- The $2N$-storage scheme with five stages and second order [11] denoted by LDD25 again constructed on the optimized five stages stability function of [7].
- The popular RK method RK4 of order four [6, p. 138] which requires at least $3N$-storage [4].

Problem 1. The linear convective wave equation for which the methods have been optimized

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0, & t > 0, \\ u(x,0) = K\exp(-x^2/c^2), \end{cases}$$

where $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, with K and c real constants and whose analytic solution is $u(x,t) = K\exp[-(x-t)^2/c^2]$. In our numerical experiments we take the same values as in [11], i.e. $K = 0.5, c = 3$.

To discretize the space variable we consider a uniform grid with nodes $x_j = j\Delta x$ for several values of Δx that will be given below. The spatial derivative is approximated by a 9 point centered difference scheme of 8th order given by

$$\frac{\partial u}{\partial x}(x_j,t) \approx \frac{1}{\Delta x}\left[-\frac{4}{5}(u_{j-1}-u_{j+1}) + \frac{1}{5}(u_{j-2}-u_{j+2}) - \frac{4}{105}(u_{j-3}-u_{j+3}) + \frac{1}{280}(u_{j-4}-u_{j+4})\right]. \tag{8}$$

The space domain extends from $x = -50$ to $x = 450$ and the time integration runs the interval $[0, 400]$ with several fixed step sizes. The values of $u(x_j, t)$ for all grid points $x_j = j\Delta x$ outside the interval $[-50, 450]$ are assumed to be zero because $u(x,t)$ is negligible for $|x - t| \geqslant 50$.

Table 3 shows the maximum of the absolute errors over the integration interval measured in the ℓ_∞ norm for the considered methods and different values of Δx and Δt. The symbols ***** indicate that the integration process is unstable. In Fig. 3, we have depicted the efficiency curves for the four methods corresponding with the stable results given in Table 3, i.e. the vertical axis shows the absolute global errors in logarithm scale ($\log_{10} | E |$) whereas the horizontal axis shows the computational cost measured by the number of function evaluations (NFCN) required by each method.

Table 3
Absolute errors in Problem 1

Δx	Δt	LDD46	New LDD46	LDD25	RK4
1/2	0.8000	*****	5.96×10^{-3}	1.70×10^{-2}	4.86×10^{-2}
1/2	0.7505	8.37×10^{-2}	4.66×10^{-3}	1.52×10^{-2}	4.17×10^{-2}
1/3	0.5000	*****	8.33×10^{-4}	6.28×10^{-3}	1.25×10^{-2}
1/3	0.4706	2.57×10^{-2}	6.55×10^{-4}	5.39×10^{-3}	1.01×10^{-2}
1/4	0.4000	*****	3.34×10^{-4}	3.53×10^{-3}	5.55×10^{-3}
1/4	0.3333	3.83×10^{-3}	1.61×10^{-4}	2.14×10^{-3}	2.74×10^{-3}
1/5	0.2701	*****	6.91×10^{-5}	1.19×10^{-3}	1.19×10^{-3}
1/5	0.2556	2.79×10^{-4}	5.54×10^{-5}	1.02×10^{-3}	9.58×10^{-4}

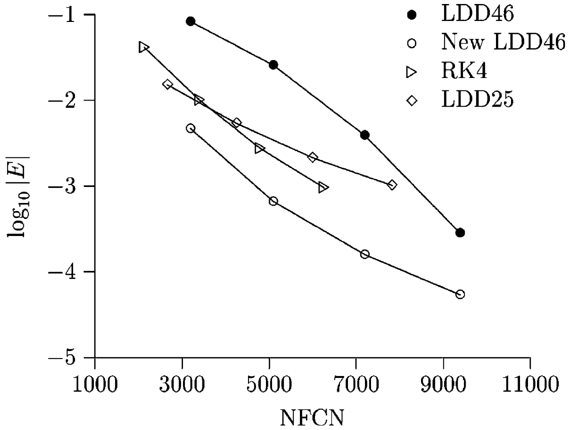

Fig. 3. Efficiency curves in Problem 1.

Problem 2. The nonlinear scalar wave model problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (1 + (x - t)u)\dfrac{\partial u}{\partial x} + \dfrac{c^2}{2}\left(\dfrac{\partial u}{\partial x}\right)^2 = 0, \quad t > 0, \\ u(x, 0) = K\exp(-x^2/c^2), \end{cases}$$

where K and c are positive real constants and whose analytic solution is $u(x,t) = K\exp(-(x-t)^2/c^2)$. In this case, the spatial domain ranges from $x = -50$ to $x = 130$ and as spatial discretization we have used the same as in the above problem. The parameter values are $K = 1/2$, $c = 2$, and the problem has been integrated up to $t_{\text{end}} = 80$. Again, Table 4 shows the maximum absolute errors obtained (in the ℓ_∞ norm) with each method in function of the step size Δt and the spatial mesh Δx, and the symbols ***** indicate that the integration process is unstable. In Fig. 4, we show the efficiency curves (only the stable integrations) of the four methods.

As can be observed from Tables 4 and 5, the error behaviour is favourable for the new LDD46 scheme. In the case of larger step sizes, the integration process for the LDD46 becomes unstable whereas the new scheme which has a larger stability interval performs properly. On the other hand, when the step size is selected so that the integration process is stable for all methods, the new scheme also performs more efficiently than the other methods (see Figs. 3 and 4).

Table 4
Absolute errors in Problem 2

Δx	Δt	LDD46	New LDD46	LDD25	RK4
1/2	0.3007	*****	8.07×10^{-3}	2.58×10^{-2}	1.94×10^{-2}
1/2	0.2666	3.84×10^{-2}	8.03×10^{-3}	1.99×10^{-2}	8.81×10^{-3}
1/3	0.1777	*****	3.35×10^{-4}	3.62×10^{-3}	2.38×10^{-3}
1/3	0.1649	4.64×10^{-3}	2.82×10^{-4}	2.81×10^{-3}	1.85×10^{-3}
1/4	0.1501	*****	7.90×10^{-5}	2.45×10^{-3}	*****
1/4	0.1201	3.98×10^{-3}	4.49×10^{-5}	1.16×10^{-3}	4.47×10^{-4}
1/5	0.1000	*****	1.24×10^{-5}	6.82×10^{-4}	2.14×10^{-4}
1/5	0.0920	1.97×10^{-4}	9.26×10^{-6}	5.10×10^{-4}	1.55×10^{-4}

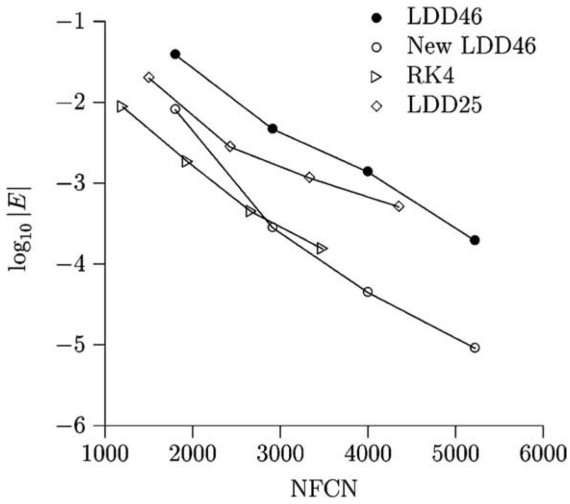

Fig. 4. Efficiency curves in Problem 2.

Problem 3. The well-known nonlinear KdV problem [5]

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

has the one-soliton solution given by

$$u(x,t) = 2\mathrm{sech}^2(x-4t),$$

where the soliton amplitude is 2 and the soliton speed is 4. We consider the temporal variable in the interval $[t_0, t_{\mathrm{end}}] = [0,2]$ and the range of the spatial variable between $[x_L, x_R] = [-20, 20]$. Note that outside this spatial range, the solution is 0 for all practical purposes.

To solve this problem we use a fourth-order modified Galerkin space discretization proposed by Frutos and Sanz-Serna [5] in the uniformly spaced grid $x_j = x_L + j(x_R - x_L)/J$, $j = 0, 1, \ldots, J$, for a given positive integer J. If $U_j(t)$ denotes the approximation to $u(x_j, t)$, these functions satisfy the differential system

$$\frac{\dot U_{j-2} + 26\dot U_{j-1} + 66\dot U_j + 26\dot U_{j+1} + \dot U_{j+2}}{120} - \frac{U_{j-2}^2 + 10U_{j-1}^2 - 10U_{j+1}^2 - U_{j+2}^2}{8\Delta x} - \frac{U_{j-2} - 2U_{j-1} + 2U_{j+1} - U_{j+2}}{2\Delta x^3} = 0, \quad (j = 0, \ldots, J). \tag{9}$$

Here it is assumed that U_{-2}, U_{-1}, U_{J+1}, U_{J+2} vanish for all values of t. For the value $J = 89$, the resulting initial-value problem is mildly-stiff and the numerical integration has been performed for several time steps Δt. We present here the results for $\Delta t = 0.025$ because for larger time steps only the new LDD46 is stable. In

Table 5
Maximum global errors in Problem 3

Schemes	$\|\cdot\|_1$	$\|\cdot\|_2$	$\|\cdot\|_\infty$
LDD46	5.2730	0.7803	0.1930
New LDD46	0.2875	0.0551	0.0300
LDD25	0.2841	0.0550	0.0304
RK4	0.2581	0.0539	0.0316

Table 5 we display the maximum global error along the time integration for the four schemes in different norms. The behaviour of the new method is comparable to the lower-order method LDD25 and it is clearly superior to LDD46. In our opinion, this is due to the fact that LDD25, in spite of its low order, possesses good stability properties and therefore for problems in which stability is an essential requirement it provides good results. On the other hand RK4 provides also a comparable accuracy but at the price of using more storage than the other schemes.

Fig. 5 shows the global errors of the numerical solution at the final time $t_{\rm end} = 2$ for the two LDD46-schemes, and Fig. 6 shows the numerical solutions obtained with both schemes. It should be remarked that the numerical solution provided by the new scheme is qualitatively more accurate than the numerical solution provided by the LDD46.

Problem 4. The first order hyperbolic system

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} - \omega u = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} + \omega v = 0, \quad 0 \leqslant x \leqslant \ell, \quad t \geqslant 0,$$

with the initial and boundary conditions

$$v(x,0) = 0, u(x,0) = a(x), \quad 0 < x \leqslant \ell, \qquad v(0,t) = u(0,t) = 0, \quad t > 0.$$

Here the initial and boundary conditions have been chosen so that the problem has the analytic solution

$$v(x,t) = a(x)\sin(\omega t), \qquad u(x,t) = a(x)\cos(\omega t),$$

with $a(x) = x(x-\ell)/\ell$.

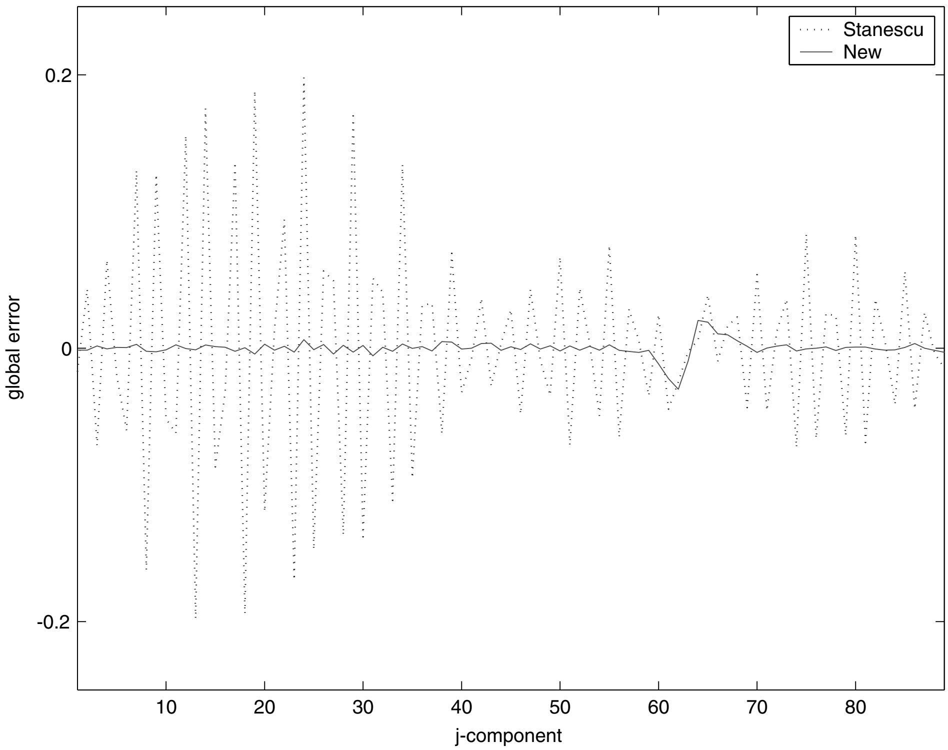

Fig. 5. Global errors in each component at $t_{\rm end} = 2$.

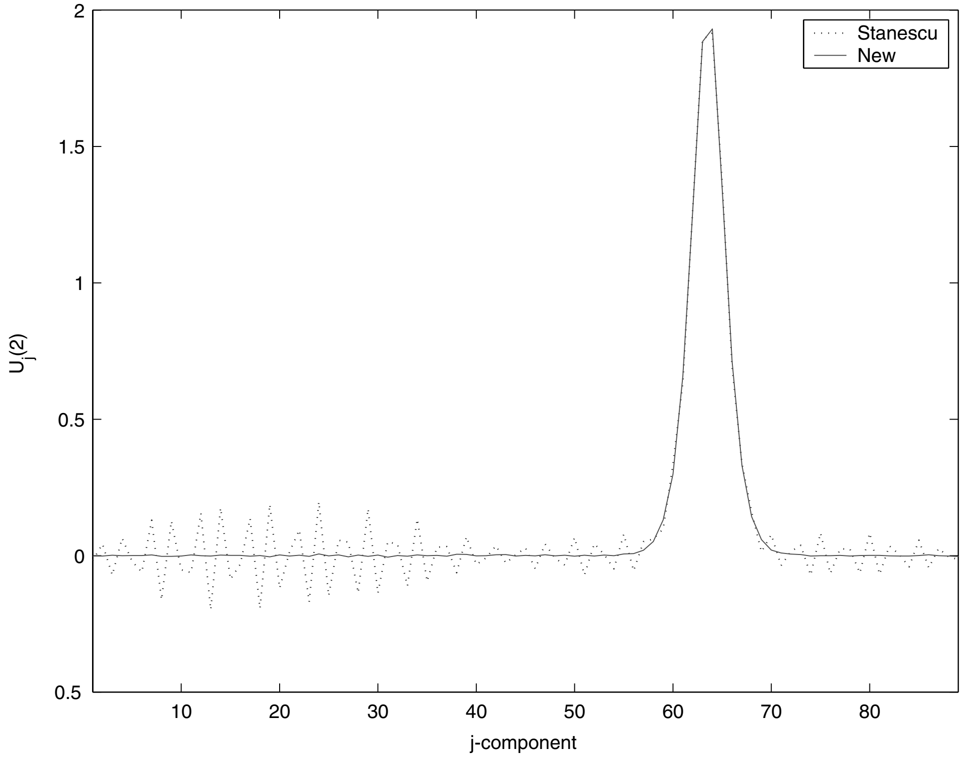

Fig. 6. Numerical solution obtained with both schemes at $t_{\text{end}} = 2$.

To solve this problem we consider an uniform grid ($\Delta x = \ell/N$) on the interval $[0, \ell]$, and we approximate the spatial derivatives by using second order symmetric differences at the internal grid points and one-sided differences at the boundary point $x = \ell$. Denoting by $v_j(t)$ and $u_j(t)$ the approximations to $v(x_j, t)$ and $u(x_j, t)$, respectively, we obtain the following differential system

$$\begin{aligned}
\dot{v}_1 &= \frac{v_1 u_2}{2\Delta x} - \frac{u_1 v_2}{2\Delta x} + \omega u_1, \\
\dot{v}_j &= v_j \frac{u_{j+1} - u_{j-1}}{2\Delta x} - u_j \frac{v_{j+1} - v_{j-1}}{2\Delta x} + \omega u_j, \quad j = 2, \ldots, N-1, \\
\dot{v}_N &= v_N \frac{u_{N-2} - 4u_{N-1} + 3u_N}{2\Delta x} - u_N \frac{v_{N-2} - 4v_{N-1} + 3v_N}{2\Delta x} + \omega u_N, \\
\dot{u}_1 &= \frac{v_1 u_2}{2\Delta x} - \frac{u_1 v_2}{2\Delta x} - \omega v_1, \\
\dot{u}_j &= v_j \frac{u_{j+1} - u_{j-1}}{2\Delta x} - u_j \frac{v_{j+1} - v_{j-1}}{2\Delta x} - \omega v_j, \quad j = 2, \ldots, N-1, \\
\dot{u}_N &= v_N \frac{u_{N-2} - 4u_{N-1} + 3u_N}{2\Delta x} - u_N \frac{v_{N-2} - 4v_{N-1} + 3v_N}{2\Delta x} - \omega v_N,
\end{aligned}$$

with the initial conditions

$$v_j(0) = 0, \quad u_j(0) = a(x_j), \quad j = 1, \ldots, N.$$

Table 6
Absolute errors in Problem 4

Δx	Δt	LDD46	New LDD46	LDD25	RK4
1/5	1/20	*****	3.15×10^{-4}	8.92×10^{-3}	*****
1/5	1/30	3.32×10^{-3}	1.52×10^{-4}	5.29×10^{-3}	2.62×10^{-3}
1/10	1/65	*****	6.88×10^{-6}	6.39×10^{-4}	1.19×10^{-4}
1/10	1/70	9.84×10^{-4}	5.91×10^{-6}	5.80×10^{-4}	1.02×10^{-4}
1/20	1/110	*****	8.37×10^{-7}	1.76×10^{-4}	*****
1/20	1/150	1.12×10^{-4}	2.42×10^{-7}	8.69×10^{-5}	4.19×10^{-6}

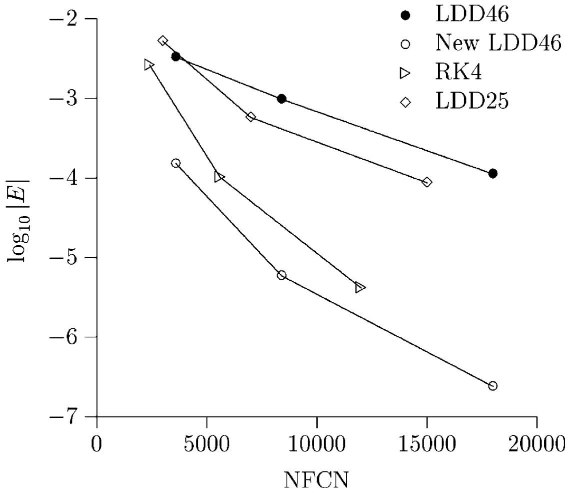

Fig. 7. Efficiency curves in Problem 4.

In our test we choose the parameter values $\ell = 50$, $\omega = 4$, and the problem has been integrated up to $t_{\text{end}} = 20$. In Table 6 we show the maximum absolute errors (in the ℓ_∞ norm) for the considered methods and different values of Δx and Δt. Again, the symbols ***** indicate that the integration process is unstable. Fig. 7 shows the efficiency curves for the four methods corresponding with the stable integrations given in Table 6. As it can be observed in this figure, our method turns out to be the most efficient of the tested methods.

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